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Void formation and growth in a class of compressible solids

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Abstract. A new class of compressible elastic solids, which includes the Blatz-Ko material as a special case, is proposed. A closed-form solution is constructed and studied for a bifurcation problem modeling void formation in this class of compressible elastic solids. The relation between the void-formation condition and the material parameters is obtained analytically. An energy comparison of the void-formation deformation and the homogeneous expansion deformation is carried out.

1. Introduction

The problem of void formation and growth in solids has given rise to many investigations in applied mechanics due to its importance in understanding damage and failure mechanisms (cf. Chu and Needleman [1], Goods and Brown [2], Gurson [3], Needleman [4], Tvergaard [5–7]). The experimental observation of void formation in vulcanized rubber was reported in 1958 (cf. Gent and Lindley [8]). After the publication of a fundamental paper by John Ball [9] in 1982 an extensive mathematical study of this problem arose (cf. Abeyaratne and Horgan [10], Antman and Negron-Marrero [11], Chung and Horgan [12], Giaquinta, Modica and Soucek [13], Horgan and Abeyaratne [14], Horgan and Pence [15, 16], James and Spector [17], Pericak-Spector and Spector [18], Podio-Guidugli, Vergara Caffarelli and Virga [19], Sivaloganathan [20,21], Stuart [22]). In Ball [9] the explicit solution for void formation in incompressible elastic solids was studied in detail, while existence theorems for the void-formation solutions were established for some compressible elastic solids satisfying certain convexity and growth conditions. In [9] the general results for compressible materials were applied to a specific material model which is a compressible version of the incompressible neo-Hookean material (cf. Section 7.6 of [9]), and the void-formation conditions were obtained in closed form (cf. P_{cr} in (7.60) and P^* in (7.61) of [9]).

In 1986 an explicit solution for void formation in a compressible elastic solid was constructed by Horgan and Abeyaratne [14]. In their paper, a plain strain bifurcation problem for a solid circular cylinder composed of the Blatz-Ko material (cf. Blatz and Ko [23]) was studied. The boundary of the circular cross-section of the cylinder was subjected to a radial stretch. A homogeneous expansion was found to bifurcate into a nonhomogeneous expansion when the prescribed radial stretch exceeded a critical value. The nonhomogeneous deformation in constructing the closed-form solution for this problem is that it requires the explicit integration of a non-linear second-order ordinary differential equation. However, it is worth mentioning that it is always possible for us to construct closed-form solutions for all the second-order ordinary differential equations governing the radially symmetric deformations in compressible hyperelastic solids. The reason is as follows. A second-order ordinary differential equation of this type will admit the one-parameter scaling group, and can be reduced to a first-order ordinary differential equation (cf. Bluman and Kumei [24]) which is equivalent to a two-dimensional Pfaffian differential equation. And further, any two-dimensional Pfaffian differential equation admits an integrating factor (cf. Von Westenholz [25]). Despite this, there is no general solution to the integrating factor of the first-order ordinary differential equation reduced from the second-order equation governing the radially symmetric deformation in compressible materials. This is because the first-order ordinary differential equation corresponding to different choices of compressible material models will admit different types of symmetries, and thus the corresponding integrating factors will differ from each other.

In this paper we study a bifurcation problem of the type considered in Horgan and Abeyaratne [14] for a general class of compressible elastic solids, which includes the Blatz-Ko material as a special case. Since this class of material models contains three parameters, it can simulate a broad spectrum of solids. We first construct the closed-form solution for the governing equation and then apply the traction-free boundary condition at the center of the circular cross-section of the cylinder to obtain the critical boundary value λ_{cr} at which the void-formation process starts. When the prescribed radial stretch $\lambda < \lambda_{cr}$, a homogeneous expansion deformation prevails in the cross-section, and the homogeneous deformation will, when $\lambda > \lambda_{cr}$, bifurcate into a nonhomogeneous deformation which creates a void in the center of the cross-section. It is found that the whole class of solids admits the void-formation deformation.

Our paper can be viewed as complementary to the work done by D.M. Haughton [26, 27]. In [26, 27] the problem of cavitation in elastic membranes was considered. While the impossibility of cavitation in incompressible membranes was examined in [26], the possibility of cavitation in compressible membranes was the subject of [27]. There are three essential differences between Haughton's works and ours. First, we are dealing with a plain-strain problem while the plane-stress problem was treated in Haughton's works. Second, our proof of the existence of the solution containing a cavity is analytical while the proofs in [27] were numerical. In particular, the relation between the material parameters and the critical value for the initiation of the cavity is constructed analytically in our paper, which is useful for us to understand the behavior of cavitation. An analytical relation of that kind was not found in [27]. And third, the strain-energy density function considered in this paper (cf. Section 3) is in a form different from that in [27].

In Section 2 the formulation of the boundary-value problem is presented. The void-formation solution for the problem will be obtained in Section 3 for a special class of materials proposed by us, and the functional relation of λ_{cr} with the material parameters is also established. In the last section, an energy comparison of the void-formation deformation and the homogeneous expansion deformation is carried out.

2. Mathematical Formulation of the Problem

In this section the mathematical formulation of the void-formation problem is presented. We consider a compressible isotropic hyperelastic cylinder in plane-strain. The cross-section of its underformed configuration is denoted by

$$D_0 = \{(R, \theta) : 0 \leq R \leq A, 0 < \Theta \leq 2\pi\}.$$

The boundary of the cross-section is subjected to a prescribed uniform radial stretch and the deformed configuration of D_0 is described by

$$D = \{(r, \theta) : r = r(R) \geq 0, \quad \theta = \Theta \forall (R, \Theta) \in D_0\}.$$

Here $r(R)$ is a function to be determined. The corresponding deformation gradient is

$$\mathbf{F} = \begin{pmatrix} F_{rR} & F_{r\Theta} \\ F_{\theta R} & F_{\theta\Theta} \end{pmatrix} = \begin{pmatrix} \mathrm{d}r/\mathrm{d}R & 0 \\ 0 & r/R \end{pmatrix}.$$

Let

$$\lambda_1 = F_{rR} = \frac{\mathrm{d}r}{\mathrm{d}R}, \quad \lambda_2 = F_{\theta\Theta} = \frac{r}{R} \tag{2.1}$$

be the principal stretches and $\Phi = \Phi(\lambda_1, \lambda_2)$ be the strain-energy density function for plane -strain deformations, then the first Piola-Kirchhoff stress can be expressed as

$$\mathbf{S} = \begin{pmatrix} S_{rR} & S_{r\Theta} \\ S_{\theta R} & S_{\theta\Theta} \end{pmatrix} = \begin{pmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{pmatrix} \tag{2.2}$$

with $\Phi_i = \partial\Phi/\partial\lambda_i \quad (i = 1, 2)$. The Cauchy stress $\mathbf{T}$ is related to the first Piola-Kirchhoff stress through $\mathbf{T} = (\det \mathbf{F})^{-1}\mathbf{S}\mathbf{F}^T$. In accordance with (2.2) $\mathbf{T}$ can be written as

$$\mathbf{T} = \begin{pmatrix} T_{rr} & T_{r\theta} \\ T_{\theta r} & T_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \Phi_1/\lambda_2 & 0 \\ 0 & \Phi_2/\lambda_1 \end{pmatrix}. \tag{2.3}$$

Due to the symmetries of the problem and the absence of body forces, the equilibrium equation $\nabla \cdot \mathbf{S} = 0$ can be reduced to

$$\frac{\mathrm{d}S_{rR}}{\mathrm{d}R} + \frac{S_{rR} - S_{\theta\Theta}}{R} = 0.$$

Combining this equation with (2.1) and (2.2), we find

$$\frac{\mathrm{d}\Phi_1}{\mathrm{d}R} + \frac{\Phi_1 - \Phi_2}{R} = 0, \tag{2.4}$$

or

$$R\frac{\mathrm{d}^2 r}{\mathrm{d}R^2} + \frac{1}{\Phi_{11}} + \frac{1}{\Phi_{11}}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}R} - \frac{r}{R}\right)\Phi_{12} + \Phi_1 - \Phi_2\right] = 0, \tag{2.5}$$

where $\Phi_{ij} = \partial^2\Phi/\partial\lambda_i\partial\lambda_j \quad (i, j = 1, 2)$. We are interested in those solutions of (2.5) which satisfy

$$r = \lambda A \quad \text{at} \quad R = A, \tag{2.6}$$

where $\lambda \geq 1$ is the prescribed circumferential stretch at the boundary of the cylinder. One solution of (2.5) satisfying (2.6) is

$$r = \lambda R, \tag{2.7}$$

which represents a homogeneous radial expansion of the cylinder. No void is created at the center of D by this homogeneous deformation, since $r(0) = 0$ is implied by (2.7). As void formation and growth are of interest, we try to look for another solution of (2.5) satisfying both (2.6) and the condition $r(0) > 0$, which is the task taken up in the next section.

3. Void-Formation Deformation

In this section we look for a solution of (2.5) satisfying both the condition (2.6) and the condition $r(0) > 0$. It is expected that T_{rr} will vanish at $R = 0$ once a void appears at the center of D, that is,

$$T_{rr} = \frac{1}{\lambda_2}\frac{d\Phi}{d\lambda_1} = 0 \quad \text{at} \quad R = 0 \tag{3.1}$$

according to (2.3). To integrate (2.5) we note that it can be written, according to (2.1), as

$$R\frac{d\lambda_1}{dR} - \phi(\lambda_1, \lambda_2) = 0 \tag{3.2}$$

with

$$\phi(\lambda_1, \lambda_2) = \frac{1}{\Phi_{11}}[(\lambda_2 - \lambda_1)\Phi_{12} + \Phi_2 - \Phi_1]. \tag{3.3}$$

Also,

$$\frac{d\lambda_2}{dR} = \frac{d}{dR}\left(\frac{r}{R}\right) = \frac{\lambda_1 - \lambda_2}{R} \tag{3.4}$$

is implied by (2.1). Putting this relation into (3.2), we obtain a first-order ordinary differential equation

$$\frac{d\lambda_1}{d\lambda_2} = \frac{\phi(\lambda_1, \lambda_2)}{\lambda_1 - \lambda_2}. \tag{3.5}$$

Motivated by the work of Horgan and Abeyaratne [14] we look for a special form of Φ such that (3.5) can be written, for some function G, as

$$\frac{d\lambda_1}{d\lambda_2} = \frac{\phi(\lambda_1, \lambda_2)}{\lambda_1 - \lambda_2} = G(t),$$

where $t = \lambda_1/\lambda_2$. The reason for this will be made clear later on. We consider a class of solids with the strain-energy density function in a form

$$W = a_1(\lambda_1^{-n} + \lambda_2^{-n} + \lambda_3^{-n}) + a_2[(\lambda_1\lambda_2)^{-n} + (\lambda_2\lambda_3)^{-n} + (\lambda_3\lambda_1)^{-n}] + h^*(J),$$

where λ_3 denotes the principal stretch in the direction perpendicular to the cross-section D_0; a_1 and a_2 are constants, n is a positive number, $J = \lambda_1\lambda_2\lambda_3$, and $h^*(\cdot)$ is an arbitrary function. This strain-energy density function coincides with that of the Blatz-Ko material if one selects $h^*(J) = \mu(J - 5/2), a_1 = \mu/2, a_2 = 0$, and $n = 2$, where μ is the shear modulus at infinitesimal deformations. For plane-strain deformations $\lambda_3 = 1$, and the strain-energy density becomes

$$\Phi = W(\lambda_1, \lambda_2, 1) = a(\lambda_1^{-n} + \lambda_2^{-n}) + h(J), \tag{3.6}$$

where $a = a_1 + a_2$ and $h(J) = a_1 + a_2J^{-n} + h^*(J)$. After some manipulations we have, in accordance with (3.3) and (3.6),

$$\frac{\phi(\lambda_1, \lambda_2)}{\lambda_1 - \lambda_2} = -\frac{an(t^{n+1} - 1)t/(t-1) + th''\lambda_2^2\lambda_1^{n+2}}{an(n+1) + h''\lambda_2^2\lambda_1^{n+2}}, \tag{3.7}$$

where h'' is the second derivative of the function h. It is easy to see that if we choose

$$h'' = bJ^{-\frac{n}{2}-2}$$

then

$$\Phi = a(\lambda_1^{-n} + \lambda_2^{-n}) + h(J) = a(\lambda_1^{-n} + \lambda_2^{-n}) + \frac{4b}{n(n+2)} J^{-\frac{n}{2}} + cJ + d, \tag{3.8}$$

where b, c, d are constants, and

$$h''\lambda_2^2\lambda_1^{n+2} = b\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{n}{2}} = bt^{\frac{n}{2}}.$$

For three-dimensional deformations, the strain-energy density function corresponding to (3.8) is

$$\begin{aligned} W &= a_1(\lambda_1^{-n} + \lambda_2^{-n} + \lambda_3^{-n}) + a_2[(\lambda_1\lambda_2)^{-n} + (\lambda_2\lambda_3)^{-n} + (\lambda_3\lambda_1)^{-n}] - a_2J^{-n} \\ &\quad + \frac{4b}{n(n+2)} J^{-\frac{n}{2}} + cJ + d - a_1. \end{aligned}$$

Putting (3.7) and (3.8) into (3.5), we obtain

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}\lambda_2} = \frac{\phi(\lambda_1, \lambda_2)}{\lambda_1 - \lambda_2} = G(t)$$

with

$$G(t) = -t\left\{\alpha t^{\frac{n}{2}} + \frac{p(t)}{n+1}\right\} / (1 + \alpha t^{\frac{n}{2}}), \tag{3.9}$$

$$p(t) = \frac{t^{n+1} - 1}{t - 1}, \tag{3.10}$$

$$\alpha = \frac{b}{an(n+1)}. \tag{3.11}$$

The differential equation can be integrated as follows. Since $t = \lambda_1/\lambda_2$ one has

$$\lambda_2 \frac{\mathrm{d}t}{\mathrm{d}\lambda_2} = \frac{\mathrm{d}\lambda_1}{\mathrm{d}\lambda_2} - t = G(t) - t$$

and, according to (3.4) and (2.1),

$$\frac{\mathrm{d}\lambda_2}{\lambda_2} = \left(\frac{\lambda_1}{\lambda_2} - 1\right)\frac{\mathrm{d}R}{R} = (t-1)\frac{\mathrm{d}R}{R},$$

$$\frac{\mathrm{d}R}{R} = \frac{\mathrm{d}R}{\mathrm{d}r}\frac{\mathrm{d}r}{r}\frac{r}{R} = \frac{\lambda_2}{\lambda_1}\frac{\mathrm{d}r}{r} = \frac{1}{t}\frac{\mathrm{d}r}{r}.$$

Combining the above relations, we get

$$\frac{\mathrm{d}t}{G(t) - t} = \frac{\mathrm{d}\lambda_2}{\lambda_2} = (t-1)\frac{\mathrm{d}R}{R} = \frac{t-1}{t}\frac{\mathrm{d}r}{r}, \tag{3.12}$$

which can be integrated to give a parametric solution,

$$R = R(t) = A\exp\int_t^{t_A} \frac{\mathrm{d}\tau}{[\tau - G(\tau)](\tau - 1)} \geq 0, \tag{3.13}$$

$$r = r(t) = \lambda A\exp\int_t^{t_A} \frac{\tau\,\mathrm{d}\tau}{[\tau - G(\tau)](\tau - 1)} \geq 0, \tag{3.14}$$

for the equation (3.5) satisfying the condition

$$R = A \quad \text{and} \quad r = \lambda A \quad \text{when} \quad t = t_A.$$

The corresponding principal stretches are

$$\lambda_1 = t\lambda_2, \quad \lambda_2 = \frac{r}{R} = \lambda\exp\int_t^{t_A} \frac{\mathrm{d}\tau}{\tau - G(\tau)}.$$

It should be noted that when the material parameter n is a rational number, the integrals in (3.13) and (3.14) may be evaluated analytically using partial fractions after an elementary substitution. When n is an even integer, no substitution is needed before we decompose the integrands into partial fractions. When $n = p^*/q^*$ for some positive integers p^* and q^*, a substitution $t = u^{2q^*}$ may be chosen. When n is an irrational number, the situation will be more complicated and one does not expect that the integrands can be decomposed into partial fractions. In this paper we do not intend to treat these different cases of n separately; instead, we shall analyze our results in a unifying manner without evaluating the integrals in (3.13) and (3.14) explicitly with respect to different values of n.

In order to study the deformation represented by the parametric solution, we define

$$s(t) = 1 + \alpha t^{\frac{n}{2}}, \quad q(t) = 1 + \frac{p(t)}{n+1} + 2\alpha t^{\frac{n}{2}}$$

such that, according to (3.9),

$$\frac{1}{t - G(t)} = \frac{s(t)}{tq(t)}.$$

We can also connect the constants a, b, c, d in the expression of $\Phi(\lambda_1, \lambda_2)$ defined in (3.8) to the shear and bulk moduli, μ and κ, by imposing the restrictions

$$\Phi(1,1) = \Phi_1(1,1) = 0, \quad \Phi_{11}(1,1) = \kappa + \frac{4}{3}\mu, \quad \Phi_{12}(1,1) = \kappa - \frac{2}{3}\mu,$$

which guarantee that the hyperelastic solids represented by Φ in (3.8) are consistent with the classical theory in the linear approximation (cf. Ogden [28, p. 349]). Invoking (3.8) and (3.11), we have

$$a = \frac{2\mu}{n^2}, \quad b = \kappa - 2\left(\frac{1}{3} + \frac{1}{n}\right)\mu, c = \frac{2}{n+2}\left(\kappa + \frac{\mu}{3}\right), d = \frac{-2}{n}\left(\kappa + \frac{\mu}{3}\right),$$

and

$$\alpha = \frac{1}{n+1}\left(\frac{n\kappa}{2\mu} - \frac{n}{3} - 1\right), 1 + \alpha = \frac{1}{n+1}\left(\frac{n\kappa}{2\mu} + \frac{2}{3}n\right) > 0.$$

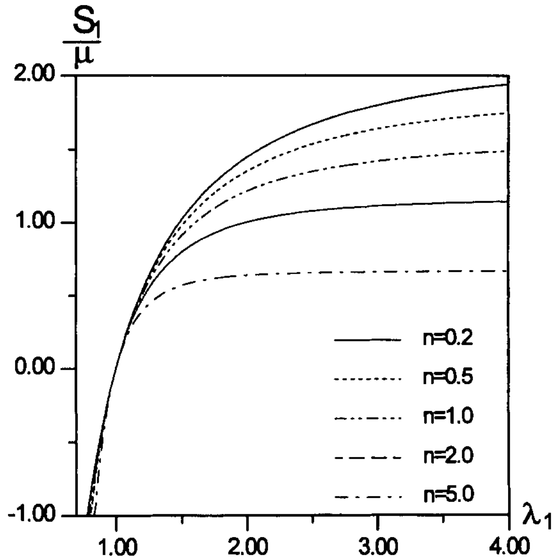

Figure 1.

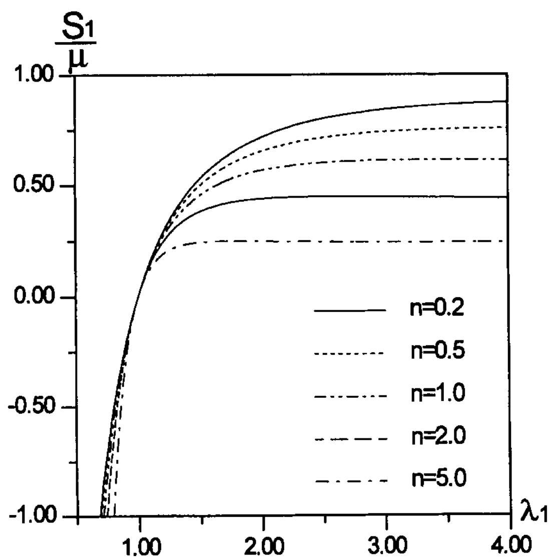

Figure 2.

Defining $S_1 = \partial\Phi/\partial\lambda_1$ as one of the principal values of the first Piola-Kirchhoff stress tensor and setting $\lambda_2 = 1$, we can have

$$S_1 = -na\lambda_1^{-n-1} - \frac{2b}{n+2}\lambda_1^{-\frac{n}{2}-1} + c$$

according to (3.8). Figs. 1 and 2 show the relation between S_1 and λ_1 with respect to different values of n, μ, and κ. Since $s(0) = 1, s(1) = 1 + \alpha$, and $s(t)$ is monotonic for $t \in (0, 1)$, we know that

$$\infty > M_1 = \sup_{t\in(0,1)} s(t) > s(t) > N_1 = \inf_{t\in(0,1)} s(t) > 0 \quad \forall t \in (0, 1).$$

Some direct computations can lead us to $p(t)/(n+1) > t^{n/2}$ for $t \in (0,1)$, and thus

$$q(t) - s(t) = \frac{p(t)}{n+1} + \alpha t^{\frac{n}{2}} > (1+\alpha)t^{\frac{n}{2}} > 0 \quad \forall t \in (0,1),$$

$$\infty > M_2 = \sup_{t\in(0,1)} q(t) > q(t) > N_2 = \inf_{t\in(0,1)} q(t) > 0 \quad \forall t \in (0,1),$$

and $t - G(t) > 0$ for $t \in (0,1)$. Since

$$\frac{1}{[G(t)-t](1-t)} = \frac{s(t)}{t(t-1)q(t)} < \frac{N_1}{-tM_2} < 0 \quad \forall t \in (0,1)$$

and, for $t_A \in (0,1)$,

$$\lim_{t\to 0^+} \int_t^{t_A} \frac{\mathrm{d}\tau}{-\tau} = \lim_{t\to 0^+} \ln\left(\frac{t}{t_A}\right) = -\infty,$$

we have

$$\lim_{t\to 0^+} R(t) = 0.$$

It is clear that,

$$R(t_A) = A \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}R(t) = \frac{R(t)}{[t-G(t)](1-t)} > 0 \quad \forall t \in (0,1),$$

and thus the circular cross section D_0 can be described completely by (3.13) with $t, t_A \in [0,1]$. Further, the deformation described by (3.13) and (3.14) is an expansion when $\lambda \geq 1$ since

$$\lambda_2 = \lambda \exp \int_t^{t_A} \frac{\mathrm{d}\tau}{\tau - G(\tau)} \geq \lambda \geq 1 \quad \forall t \in (0, t_A], t_A \in (0,1).$$

Similarly,

$$0 > \frac{N_1}{(t-1)M_2} > \frac{t}{[G(t)-t](1-t)} = \frac{s(t)}{(t-1)q(t)} > \frac{M_1}{(t-1)N_2} \quad \forall t \in (0,1) \tag{3.15}$$

and

$$\lim_{t\to 0^+} \int_t^{t_A} \frac{\mathrm{d}\tau}{\tau - 1} = \ln(1-t_A) > -\infty \quad \forall t_A \in (0,1)$$

give us that

$$\lim_{t\to 0^+} r(t) = \lambda A \exp\left[\lim_{t\to 0^+} \int_t^{t_A} \frac{\tau\,\mathrm{d}\tau}{[G(\tau)-\tau](1-\tau)}\right] > 0 \quad \forall t_A \in (0,1).$$

Combining the above relation with $R(0^+) = 0$, we may show that a void appears at the center of the deformed configuration of D_0. Due to (3.15) and

$$\lim_{t_A\to 1^-} \lim_{t\to 0^+} \int_t^{t_A} \frac{\mathrm{d}\tau}{\tau - 1} = \lim_{t_A \to 1^-} \ln(1-t_A) = -\infty,$$

we have

$$0 \leq \lim_{t_A \to 1^-} \lim_{t \to 0^+} r(t) < \lambda A \exp\left(\frac{N_1}{M_2} \lim_{t_A \to 1^-} \lim_{t \to 0^+} \int_t^{t_A} \frac{\mathrm{d}\tau}{\tau - 1}\right) = 0,$$

which, together with the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t_A} \lim_{t \to 0^+} r(t) = \frac{-t_A}{[t_A - G(t_A)](1 - t_A)} \lim_{t \to 0^+} r(t) < 0,$$

suggests that the onset of the formation of the void is in the limit $t_A \to 1^-$. Since we only aim at the void-formation deformation, it is sufficient to consider $t, t_A \in (0, 1)$. Now we want to apply the traction-free boundary condition (3.1) to obtain the critical value $\lambda = \lambda_{cr}$ at which the void-formation process starts. From (2.2) and (3.8) we obtain

$$T_{rr} = \frac{S_{rR}}{\lambda_2} = -nat^{-n-1}\lambda_2^{-n-2} - \frac{2b}{n+2} t^{-\frac{n}{2}-1}\lambda_2^{-n-2} + c.$$

In evaluating T_{rr} in the limit $t \to 0$, it will be found that l'Hospital's rule is useless and some special treatment is needed. Noting that

$$t^{-n-1} = t_A^{-n-1} \exp\left[(n+1)\int_t^{t_A} \frac{\mathrm{d}\tau}{\tau}\right] \quad \text{and} \quad t^{-\frac{n}{2}-1} = t_A^{-\frac{n}{2}-1} \exp\left[\left(\frac{n}{2}+1\right)\int_t^{t_A} \frac{\mathrm{d}\tau}{\tau}\right],$$

we can write

$$t^{-n-1}\lambda_2^{-n-2} = t_A^{-n-1}\lambda^{-n-2} \exp \int_t^{t_A} \frac{(n+1)G(\tau) + \tau}{\tau[G(\tau) - \tau]} \mathrm{d}\tau,$$

$$t^{-\frac{n}{2}-1}\lambda_2^{-n-2} = t_A^{-\frac{n}{2}-1}\lambda^{-n-2} \exp\left[-\left(\frac{n}{2}+1\right)\int_t^{t_A} \frac{G(\tau) + \tau}{\tau[\tau - G(\tau)]} \mathrm{d}\tau\right].$$

Since for $t_A \in (0, 1)$ and $t \in (0, t_A]$

$$\frac{t + G(t)}{t[t - G(t)]} = \frac{n + 1 - p(t)}{(n+1)tq(t)} \geq \frac{n + 1 - p(t_A)}{(n+1)tM_2} > 0, \text{ and } \lim_{t \to 0^+} \int_t^{t_A} \frac{d\tau}{\tau} = \infty,$$

we know that

$$\lim_{t \to 0^+} t^{-\frac{n}{2}-1}\lambda_2^{-n-2} = 0.$$

A direct computation renders

$$j(t) = \frac{(n+1)G(t) + t}{t[G(t) - t]} = \frac{\alpha n t^{\frac{n}{2}-1} + p(t) - t^n}{q(t)}, \tag{3.16}$$

which is obviously bounded for $t \in (0, t_A]$, and thus

$$\lim_{t \to 0^+} t^{-n-1}\lambda_2^{-n-2} \in (0, \infty).$$

As a result,

$$\lim_{t \to 0^+} T_{rr} = -na\lambda^{-n-2}t_A^{-n-1} \exp \int_0^{t_A} j(\tau)\, \mathrm{d}\tau + c = 0,$$

or

$$\frac{c}{na}\lambda^{n+2} = t_A^{-n-1} \exp \int_0^{t_A} j(\tau)\,\mathrm{d}\tau = v(t_A). \tag{3.17}$$

Since $r(0) = 0$ in the limit $t_A \to 1^-$, one can define

$$\lambda_{cr} = (nav(1)/c)^{\frac{1}{n+2}} \tag{3.18}$$

as the critical value of λ. A void will be formed at the center of the cross-section D_0 when $\lambda = \lambda_{cr}$. It is easy to see that

$$\lim_{t_A \to 0+} v(t_A) = \infty, \quad v(1) \in (0, \infty).$$

Some manipulations give

$$\frac{\mathrm{d}}{\mathrm{d}t_A} v(t_A) = -\frac{v(t_A)g(t_A)}{t_A q(t_A)},$$

where

$$g(t_A) = (n+1)q(t_A) - \alpha n t_A^{\frac{n}{2}} - t_A \frac{1 - t_A^n}{1 - t_A} = n + 1 + \alpha(n+2)t_A^{\frac{n}{2}} + 1,$$

which is positive because $g(t_A)$ is monotonic for $t_A \in (0,1)$ and $g(0) = n+2, g(1) = 1 + \alpha > 0$. It is obvious that $\frac{\mathrm{d}}{\mathrm{d}t_A} v(t_A) < 0$ for $t \in (0,1)$. As a result, given a value of $\lambda \geq \lambda_{cr}$, we can solve Eq. (3.17) for a unique $t_A \in (0,1)$, and the deformation is described by (3.13) and (3.14). One can see that $r(0^+)$ is proportional to λ and the void formed will be enlarged with respect to λ. For a given $\lambda < \lambda_{cr}$, (3.17) cannot give us a $t_A \in (0,1)$, and the parametric solution (3.13) and (3.14) cannot describe the deformation of the cross-section D_0. In this situation, the homogeneous expansion described by (2.7) prevails in D_0.

We would also like to study the relation between λ_{cr} and the material parameters n, μ, and κ. Combining (3.16)–(3.18), we arrive at

$$\lambda_{cr} = \left[\frac{na}{c} \exp \int_0^1 \frac{(n+1)G(t) + t}{t[G(t) - t]} \mathrm{d}t \right]^{\frac{1}{n+2}}.$$

Since

$$\frac{a}{c} = \frac{n+2}{n^2} \left(\frac{\kappa}{\mu} + \frac{1}{3} \right)^{-1}$$

and with the value of $G(t)$ depending on the parameter

$$\alpha = \frac{b}{an(n+1)} = \frac{n\kappa/\mu - 2(n/3 + 1)}{2(n+1)},$$

we see that it is the ratio of κ and μ, instead of their absolute values, which affects the value of λ_{cr}. In other words, two materials with the same values of n and κ/μ will have the same λ_{cr}, even if their shear and bulk moduli are different. The functional relation between λ_{cr} and n and $K = \kappa/\mu$ is complicated and is plotted in Fig. 3. Corresponding to a given value of

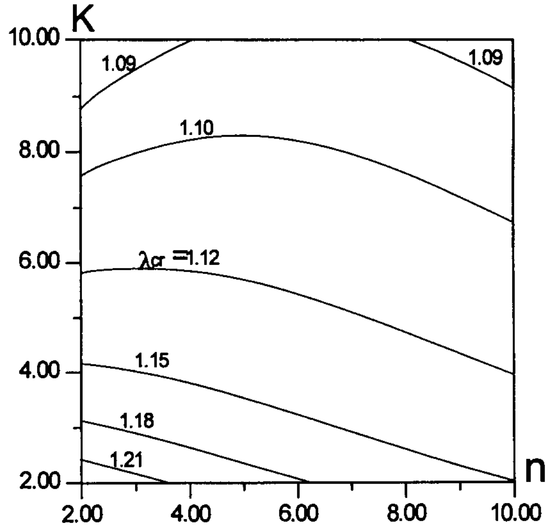

Figure 3.

λ_{cr}, there are infinitely many combinations of n and K. We can also see that λ_{cr} decreases as n or K increases.

Horgan and Abeyaratne [14] carried out an analysis to show that the void-formation problem may be viewed as providing an idealized model describing the growth of a pre-existing micro-void. We do not intend to pursue this issue here, but put out attention on the energy comparison of the void-formation deformation and the homogeneous expansion deformation, which will be the subject of the next section.

4. Energy Comparison

The potential energies for the homogeneous deformation and the void-formation deformation will be compared in this section. It will be shown that for $\lambda > \lambda_{cr}$, the potential energy associated with the void-formation deformation will always be lower than that with the homogeneous expansion deformation, and thus the void-formation deformation is relatively stable. We first define the potential energy per unit length of the cylinder as

$$E = \int_{\Omega} \Phi \, \mathrm{d}\Omega = 2\pi \int_0^A R\Phi \, \mathrm{d}R, \tag{4.1}$$

where Ω is the area of the cross section D_0. Differentiating (2.4) and adopting (2.1), (3.4), and

$$\frac{\mathrm{d}\Phi}{\mathrm{d}R} = \Phi_1 \frac{\mathrm{d}\lambda}{\mathrm{d}R} + \Phi_2 \frac{\mathrm{d}\lambda}{\mathrm{d}R},$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}R}\left\{R^2\left[\Phi - (\lambda_1 - \lambda_2)\frac{\partial\Phi}{\partial\lambda_1}\right]\right\} = 2R\Phi,$$

which, together with (4.1), reveals that

$$E = \pi A^2 \left[\Phi(\lambda_1^A, \lambda_2^A) - (\lambda_1^A - \lambda_2^A)\frac{\partial\Phi}{\partial\lambda_1^A}(\lambda_1^A, \lambda_2^A)\right],$$

where λ_1^A, λ_2^A denote the principal stretches at $R = A$. For the homogeneous deformation we have $\lambda_1^A = \lambda_2^A = \lambda$, and the corresponding potential energy is

$$E_h = E(\lambda, \lambda) = \pi A^2 \left[\Phi(\lambda_1, \lambda_2) = \pi A^2 \left[\left(2a + \frac{4b}{n(n+2)}\right) \lambda^{-n} + c\lambda^2 + d\right] .\right.$$

We have applied (3.8) in obtaining the last expression. For the void-formation deformation $\lambda_2^A = \lambda$ and $\lambda_1^A = t_A\lambda$, the potential energy can be obtained, after some manipulations, as

$$E_c = E(t_A\lambda, \lambda) = \pi A^2[m(\lambda)\lambda^{-n} + c\lambda^2 + d],$$

where

$$m(\lambda) = a + a(1 + n - nt_A^{-1})t_A^{-n} + \left(\frac{2}{n} + 1 - t_A^{-1}\right) \frac{2b}{n+2} t_A^{-\frac{n}{2}}.$$

The difference between E_h and E_c can be expressed as

$$E_h - E_c = \pi A^2 \lambda^{-n} U,$$

where

$$U = \frac{2\mu}{n^2}[1 - (1 + n - nt_A^{-1})t_A^{-n}] + \frac{4b}{n(n+2)} t_A^{-\frac{n}{2}-1} \left[t_A^{\frac{n}{2}+1} - \left(1 + \frac{n}{2}\right) t_A + \frac{2}{n}\right].$$

Since $\kappa > 2\mu/3$ and $n > 0$, the relation $b = \kappa - 2\mu(1/3 + 1/n)$ implies that $b > -2\mu/n > -\mu(n+2)/n$, which in turn implies

$$U > \frac{2\mu}{n^2} t_A^{-\frac{n}{2}-1} S(t_A),$$

where

$$S(t_A) = -t_A^{\frac{n}{2}+1} - (n+1)t_A^{-\frac{n}{2}+1} + nt_A^{-\frac{n}{2}} + (n+2)t_A - n.$$

It is easy to prove that

$$\lim_{t_A \to 0+} S(t_A) > 0, S(1) = 0,$$

and $S(t_A)$ is monotonic for $t \in (0, 1)$. Thus $S(t_A) > 0$ for $t_A \in (0, 1)$ and $E_h - E_c > 0$.

5. Concluding Remarks

Even though only one special case (*i.e.*, the Blatz-Ko material) of our material model (cf. (3.8)) has been related to experimental data, we conjecture that the results obtained in this paper might still be of practical use, since there might be materials with chemical compositions similar (but not equal) to those of Blatz-Ko material. The mechanical behaviours of such kind of materials might not be appropriately described by the Blatz-Ko material model, while they might still be described by our material model.

One major issue in the work of Horgan and Abeyaratne [14] is to show that the void-formation problem may be viewed as providing an idealized model describing the growth of a pre-existing micro-void. In this paper we have not pursued this issue. This is because an investigation of this issue for our material model, which can be viewed as a generalization of the Blatz-Ko material model, may not help us to gain any conceptually new insight into the void-formation problem. In fact, the issue with which we are concerned in this paper is quite different from that of Horgan and Abeyaratne [14], since we are interested in clarifying how the void-formation condition depends on different material properties. That issue was not dealt with in the paper written by Horgan and Abeyaratne.

Finally, the parametric solution studied in this paper is defined for the parameter t ranging from zero to one. It has been shown in this paper that it is sufficient to consider the parameter in this range in order to study the void-formation deformation. However, one might also wish to know what the deformations as described by (3.13) and (3.14) would look like for values of the parameter t outside the zero-to-one range. This will be the subject of a future study.

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